Experimental Uncertainties

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Introduction

- When experimenters do experiments, it is critical that they provide an estimate of their experimental uncertainty.
  - Experimental results should always be presented in the form:

$$m \pm \sigma_m$$

  - Here, σ_m is the experimenter's assessment of the precision of his or her result based on knowledge of the experimental apparatus, data and method.
  - Note: Some students assume that σ_m is the difference between their result and truth. That is mistaken.

Why Uncertainties Matter

- To illustrate why this is important, suppose a physicist is searching for a **new force** by measuring its strength α. If the force exists, α will be non-zero.
  - Imagine first that she finds
    $$\alpha = 0.5 \pm 0.1$$
    In this case, the deviation from zero is much larger than her uncertainty, hence one concludes that
    there is a new force.
  - Now imagine that she finds
    $$\alpha = 0.5 \pm 0.6$$
    In this case, the deviation from zero is within the experimental uncertainties, so the conclusion is
    there is no evidence for a new force.
- Notice that the central values of the two results are identical, yet the conclusions are dramatically different. The conclusions hinge on her estimate of the experimental uncertainty.
- Today: how to figure out your experimental uncertainty.

Types of uncertainty

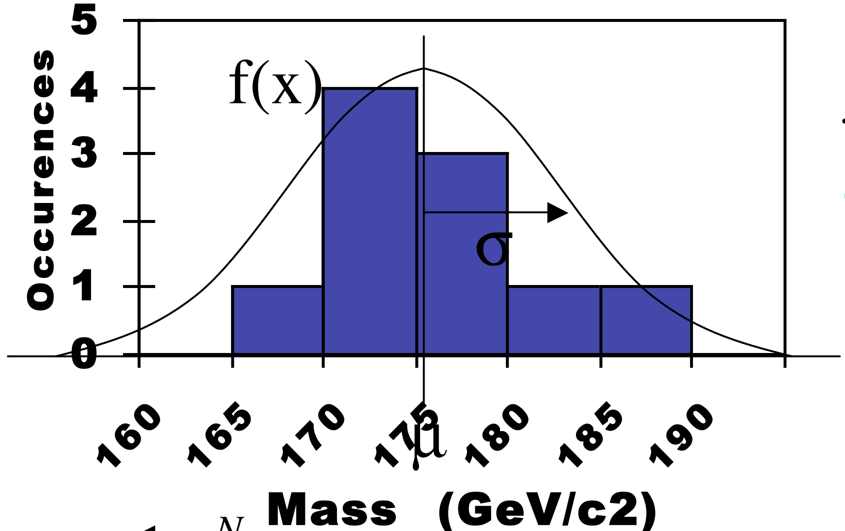
- **Random uncertainties**
  These are uncertainties that produce scatter among repeated measurements.
  - Determining random uncertainties
    - Estimate the reproducibility of your readings. Example: Rulers, thermometers, etc, typically 1/5 the finest division on the scale
    - Make repeat measurements (see next slide)

- **Systematic Uncertainties**
  These are uncertainties that shift ALL your measurements in the same direction.
  - Examples:
    - Clock speed for counting experiments
    - Meter calibration
    - Unknown Zero offsets

Random Errors

- Suppose that you have discovered a new quark and you make a series of measurements of its mass:

	Mass x (GeV/c2)
1	175
2	177
3	168
4	182
5	174
6	178
7	170
8	174
9	186
10	173



f(x) is parent distribution

$$m = \frac{1}{N}\sum_{i=1}^{N} x_i = 175.7\ \mathrm{GeV/c^2}$$

m is best estimate of μ

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - m)^2$$

s^2 is variance.
s is best estimate of σ

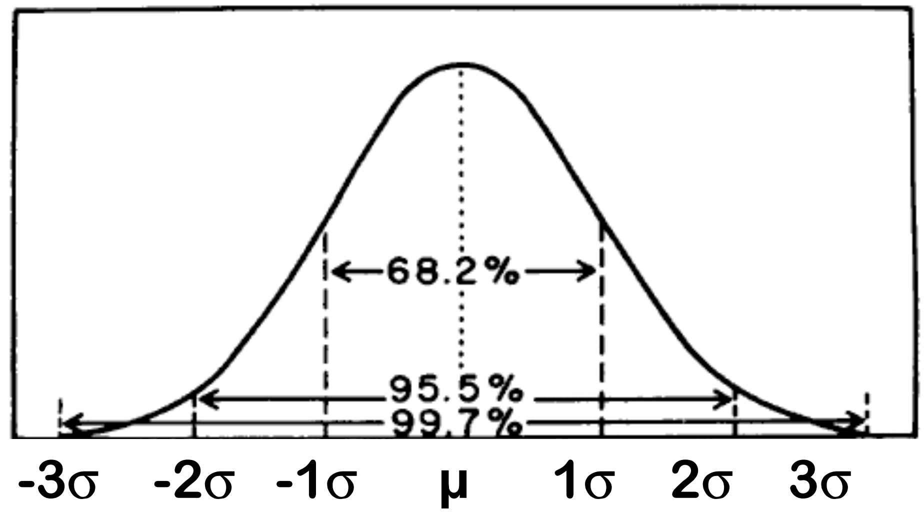
$$\sigma_m = s/\sqrt{N} = 1.7\ \mathrm{GeV/c^2}$$

σ_m is best estimate of $|m - \mu|$

The Gaussian Distribution

- Gaussian distribution = Bell curve = Normal distribution

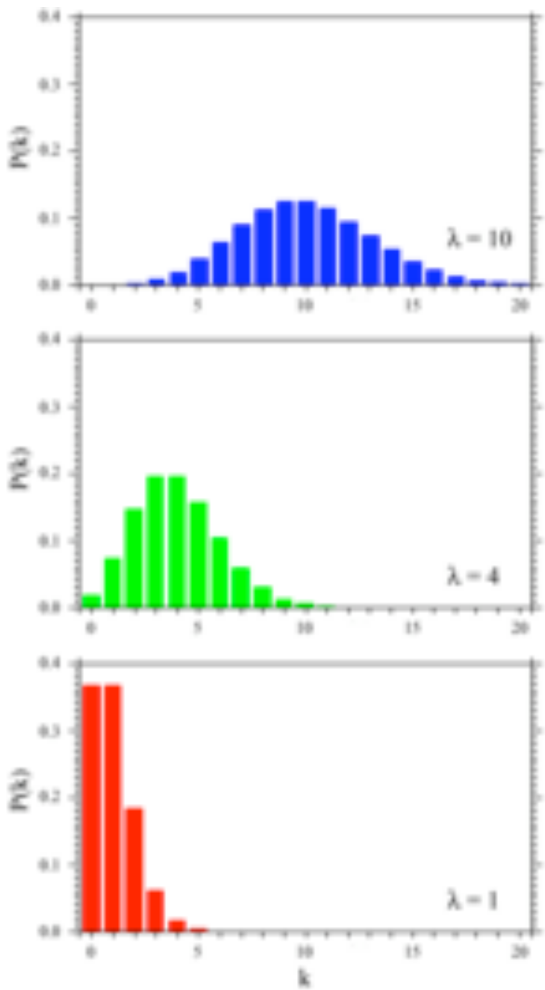
$$f(x)\Delta x = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}} \Delta x$$



- **$f(x)\Delta x$ = prob that a m'ment will fall between x and $x+\Delta x$.**
- **Gaussians: Counting experiments.**
  - **Ex: number of gamma rays emitted in a 1 minute interval.**
  - **Best estimates: m = N and σ_m = sqrt(N)**
  - **Caveat: if the number of counts is less than ~20, they will follow not a gaussian, but a Poisson distribution**
- **Gaussians: elsewhere**
  Central Limit Thm: averages are gaussian distributed, even if the individual measurements are not
- **When in doubt, assume you've got a gaussian**

Poisson Distribution

- **Number of counts, when the number of counts is small.**



$$P_{\lambda}(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

λ Is mean of parent distribution
$P_{\lambda}(k)$ is probability of observing k counts

$$\sum_{k=1}^{\infty} P_{\lambda}(k) = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

Random Errors: Unequal resolutions

- We have seen that when all measurements have the same uncertainty, their mean is the best measure of truth.
- What do you do when the measurement uncertainties vary?

	x_i	σ_i
1	33	2
2	35	1
3	36	2
4	39	3
5	35	1
6	34	1

$$m = \sum_{i=1}^{N} w_i x_i \qquad w_i = \frac{1/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}$$

$$\sigma_m^2 = \frac{1}{\sum_{i=1}^{N} 1/\sigma_i^2}$$

$m \pm \sigma_m = 34.8 \pm 0.5$

Systematic Uncertainties

- These are uncertainties that shift ALL your measurements in the same direction.
  - Clock speed for counting experiments
  - Meter calibration
  - Unknown Zero offset of an angle measurements

- Determining systematic uncertainties can be hard:
  - Sometimes these are provided by manufacturer
  - Sometimes you can devise a way to measure them
  - Sometimes you have to make an educated guess.

Propagation of Errors

- **What to do when your result depends on several variables, each of which is uncertain.**

$$y = f(a,b)$$

$$\delta y = \frac{df}{da}\delta a + \frac{df}{db}\delta b$$

$$(\delta y)^2 = \left(\frac{df}{da}\right)^2 (\delta a)^2 + \left(\frac{df}{db}\right)^2 (\delta b)^2 + 2\left(\frac{df}{da}\right)\left(\frac{df}{db}\right)\delta a \delta b$$

$$\delta y_i = y_i - y_0 \approx y_i - \bar{y}$$

$$\delta a_i = a_i - a_0 \approx a_i - \bar{a}$$

$$\delta b_i = b_i - b_0 \approx b_i - \bar{b}$$

$$\frac{1}{N-1}\sum_i (\delta y)^2 = \frac{1}{N-1}\sum_i \left[\left(\frac{df}{da}\right)^2 (\delta a_i)^2 + \left(\frac{df}{db}\right)^2 (\delta b_i)^2 + 2\left(\frac{df}{da}\right)\left(\frac{df}{db}\right)\delta a_i \delta b_i\right]$$

$$\sum_i (a_i - \bar{a}) = \sum_i (b_i - \bar{b}) = 0$$

$$\sigma_y{}^2 = \left(\frac{df}{da}\right)^2 \sigma_a{}^2 + \left(\frac{df}{db}\right)^2 \sigma_b{}^2$$

Error propagation - examples

$$\sigma_y = \sqrt{\left(\frac{df}{da}\right)^2 \sigma_a{}^2 + \left(\frac{df}{db}\right)^2 \sigma_b{}^2 + \left(\frac{df}{dc}\right)^2 \sigma_c{}^2 + \ldots}$$

- Examples:
  - Sum of two variables
    $s = x_1 + x_2$
    $\sigma_s=(\sigma_{x1}{}^2 + \sigma_{x2}{}^2)^{1/2}$
  - Average of N variables
    $y=(x_1 + x_2 + \ldots + x_{N)})/N$
    $\sigma_y = [(\sigma_{x1}/N)^2 + (\sigma_{x2}/N)^2 + \ldots + (\sigma_{xN}/N)^2]^{1/2}$
    $= \sigma_x/\mathrm{sqrt}(N)$ (if $\sigma_x = \sigma_{x1}= \sigma_{x2}=\ldots=\sigma_{xN}$)
  - Product of two variables
    $d = vt$
    $\sigma_d=(t^2\sigma_v{}^2 + v^2\sigma_t{}^2)^{1/2}$
    $\sigma_d/d = (\sigma_v{}^2/v^2 + \sigma_t{}^2/t^2)^{1/2}$

Conclusions: interpreting unc.

- All results must be reported in the form $m \pm \sigma_m$
  - σ_m is your estimate of the precision of your measurement. It is based on your knowledge of your data, equipment and method
- Comparison with "truth"
  - Can be done if you are measuring something whose value is known.
  - If you have calculated your uncertainties correctly
    - 68% of the time, "truth" will lie within the range $m \pm \sigma_m$
    - 95% of the time, "truth" will lie within $m \pm 2\sigma_m$.
    - If "truth" lies outside this range, there was probably an experimental bias or source of random fluctuations that was not accounted for in your uncertainties. If this happens to you then your report should offer your thoughts on what this might be.

A Final Word on Digits

- Note:
  - We don't know the uncertainty exactly.
    - We can only estimate the σ of the parent distribution.
    - We can only estimate systematic uncertainties.
  - Even when possible, determining the uncertainty to better than ~5% of itself generally isn't useful.
    0.50 +- 0.11 is no more informative than 0.5 +- 0.1 .
  - Rule of thumb: Quote experimental uncertainties to 1 digit most of the time, and never to more than 2 digits.
- The central value, m, should be specified to the same decimal place as the uncertainty.
  - Correct: 176 ± 2
  - Wrong: 175.7 ± 2 176 ± 2.2 175.73 ± 2.21
- If you need to, use scientific notation:
  123 ± 63 meters NO (why not?)
  $(1.2 \pm 0.6) \times 10^2$ meters YES